5.

Deflection of Beam

Theory at a Glance (for IES, GATE, PSU)

5.1 Introduction

- We know that the axis of a beam deflects from its initial position under action of applied forces.
- In this chapter we will learn how to determine the elastic deflections of a beam.

Selection of co-ordinate axes

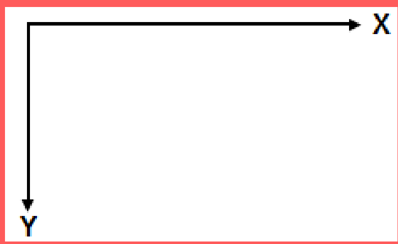
We will not introduce any other co-ordinate system. We use general co-ordinate axis as shown in the figure. This system will be followed in deflection of beam and in shear force and bending moment diagram. Here downward direction will be negative i.e. negative Y-axis. Therefore downward deflection of the beam will be treated as negative.

To determine the value of deflection of beam subjected to a given loading where we will use the formula, $EI\dfrac{d^2y}{dx^2} = M_x$.

We use above Co-ordinate system

Some books fix a co-ordinate axis as shown in the following figure. Here downward direction will be positive i.e. positive Y-axis. Therefore downward deflection of the beam will be treated as positive. As beam is generally deflected in downward directions and this co-ordinate system treats downward deflection is positive deflection.

To determine the value of deflection of beam subjected to a given loading where we will use the formula, $EI\dfrac{d^2y}{dx^2} = -M_x$.



Some books use above co-ordinate system

Why to calculate the deflections?

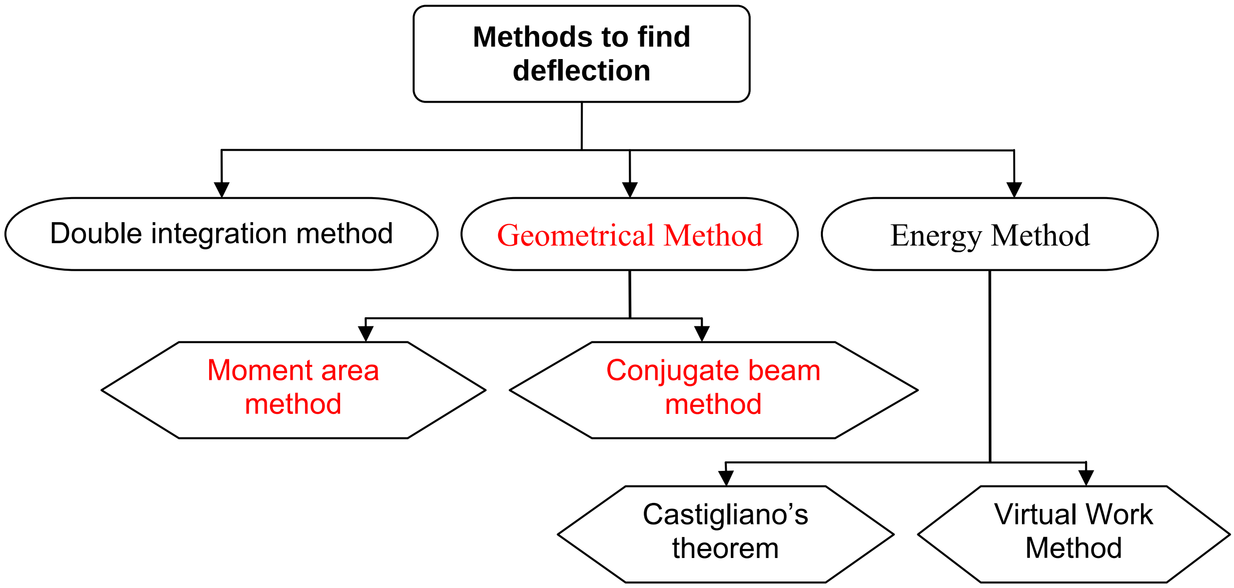
- To prevent cracking of attached brittle materials
- To make sure the structure not deflect severely and to "appear" safe for its occupants

- To help analyzing statically indeterminate structures
- Information on deformation characteristics of members is essential in the study of vibrations of machines

Several methods to compute deflections in beam

- Double integration method (*without* the use of singularity functions)
- Macaulay's Method (*with* the use of singularity functions)
- Moment area method
- Method of superposition
- Conjugate beam method
- Castigliano's theorem
- Work/Energy methods

Each of these methods has particular advantages or disadvantages.



Assumptions in Simple Bending Theory

- Beams are initially straight
- The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions
- The stress-strain relationship is linear and elastic
- Young's Modulus is the same in tension as in compression

- Sections are symmetrical about the plane of bending
- Sections which are plane before bending remain plane after bending

Non-Uniform Bending

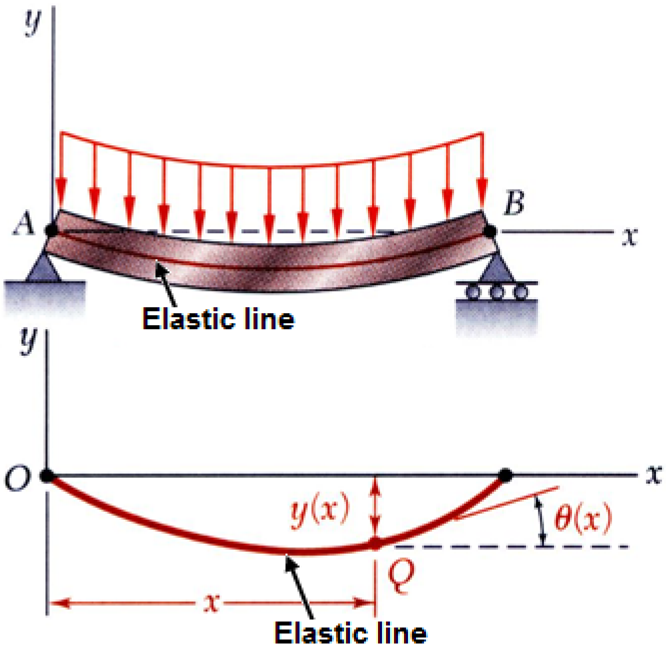
- In the case of non-uniform bending of a beam, where bending moment varies from section to section, there will be shear force at each cross section which will induce shearing stresses
- Also these shearing stresses cause warping (or out-of plane distortion) of the cross section so that plane cross sections do not remain plane even after bending

5.2 Elastic line or Elastic curve

We have to remember that the differential equation of the elastic line is

$$EI\frac{d^2y}{dx^2}=M_x$$

Proof: Consider the following simply supported beam with UDL over its length.



From elementary calculus we know that curvature of a line (at point Q in figure)

$$\frac{1}{R}=\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}$$ where $R =$radius of curvature

For small deflection, $\dfrac{dy}{dx}\approx 0$

or $\dfrac{1}{R}\approx\dfrac{d^2y}{dx^2}$

Bending stress of the beam (at point Q)

$$\sigma_x=\frac{-(M_x).y}{EI}$$

From strain relation we get

$$\frac{1}{R}=-\frac{\varepsilon_x}{y} \text{ and } \varepsilon_x=\frac{\sigma_x}{E}$$

$$\therefore \quad \frac{1}{R}=\frac{M_x}{EI}$$

Therefore $\dfrac{d^2y}{dx^2}=\dfrac{M_x}{EI}$

or $EI\dfrac{d^2y}{dx^2}=M_x$

5.3 General expression

From the equation $EI\dfrac{d^2y}{dx^2}=M_x$ we may easily find out the following relations.

- $EI\dfrac{d^4y}{dx^2}=-\omega$ Shear force density (Load)
- $EI\dfrac{d^3y}{dx^3}=V_x$ Shear force
- $EI\dfrac{d^2y}{dx^2}=M_x$ Bending moment
- $\dfrac{dy}{dx}=\theta=\text{slope}$
- y = δ = Deflection, Displacement
- Flexural rigidity = *EI*

5.4 Double integration method (*without* the use of singularity functions)

- $V_x = \int -\omega dx$
- $M_x = \int V_x dx$
- $EI\dfrac{d^2y}{dx^2} = M_x$
- $\theta = Slope = \dfrac{1}{EI}\int M_x dx$
- $\delta = Deflection = \int \theta dx$

4-step procedure to solve deflection of beam problems by double integration method

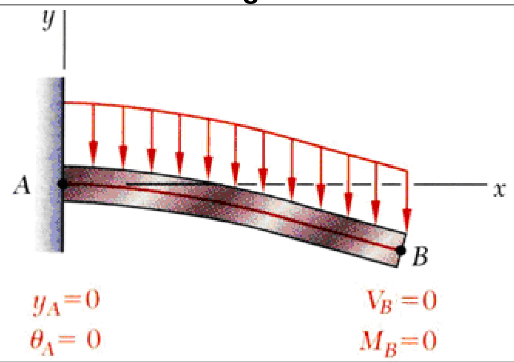
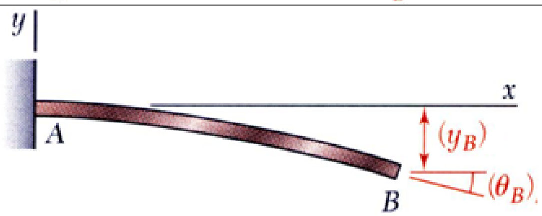
Step 1: Write down boundary conditions (Slope boundary conditions and displacement boundary conditions), analyze the problem to be solved

Step 2: Write governing equations for, $EI\dfrac{d^2y}{dx^2} = M_x$

Step 3: Solve governing equations by integration, results in expression with unknown integration constants

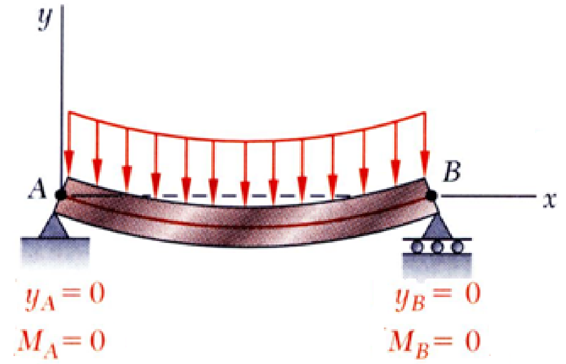
Step 4: Apply boundary conditions (determine integration constants)

Following table gives boundary conditions for different types of support.

Types of support and Boundary Conditions	Figure
Clamped or Built in support or Fixed end : (Point A) $Deflection, (y) = 0$ $Slope, (\theta) = 0$ $Moment, (M) \neq 0 \quad i.e.$ A finite value	
Free end: (Point B) $Deflection, (y) \neq 0 \quad i.e.$ A finite value $Slope, (\theta) \neq 0 \quad i.e.$ A finite value $Moment, (M) = 0$	

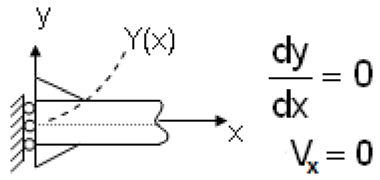
Roller (Point B) **or Pinned Support** (Point A) **or Hinged or Simply supported.**

$$Deflection, (y) = 0$$
$$Slope, (\theta) \neq 0 \quad i.e. \text{ A finite value}$$
$$Moment, (M) = 0$$



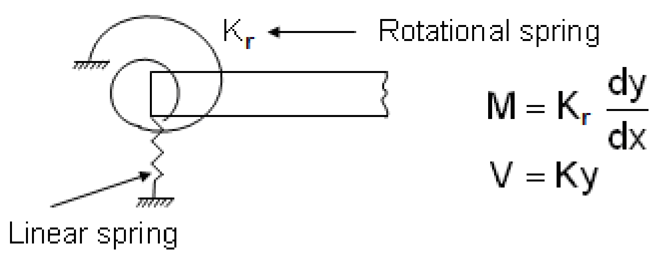
End restrained against rotation but free to deflection

$$Deflection, (y) \neq 0 \quad i.e. \text{ A finite value}$$
$$Slope, (\theta) = 0$$
$$Shear\ force, (V) = 0$$



Flexible support

$$Deflection, (y) \neq 0 \quad i.e. \text{ A finite value}$$
$$Slope, (\theta) \neq 0 \quad i.e. \text{ A finite value}$$
$$Moment, (M) = k_r \frac{dy}{dx}$$
$$Shear\ force, (V) = k.y$$



Using double integration method we will find the deflection and slope of the following loaded beams one by one.

(i) A Cantilever beam with point load at the free end.

(ii) A Cantilever beam with UDL (uniformly distributed load)

(iii) A Cantilever beam with an applied moment at free end.

(iv) A simply supported beam with a point load at its midpoint.

(v) A simply supported beam with a point load NOT at its midpoint.

(vi) A simply supported beam with UDL (Uniformly distributed load)

(vii) A simply supported beam with triangular distributed load (GVL) gradually varied load.

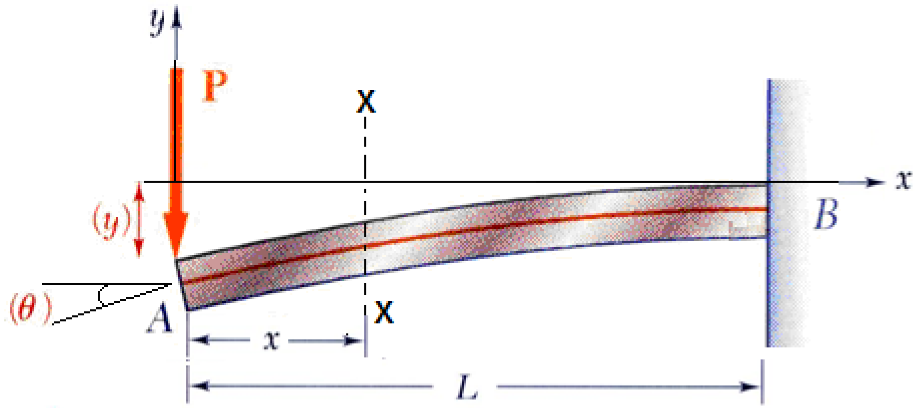
(viii) A simply supported beam with a moment at mid span.

(ix) A simply supported beam with a continuously distributed load the intensity of which at any point 'x' along the beam is $W_x = w\sin\left(\frac{\pi x}{L}\right)$

(i) A Cantilever beam with point load at the free end.

We will solve this problem by double integration method. For that at first we have to calculate (M_x).

Consider any section XX at a distance 'x' from free end which is left end as shown in figure.



$\therefore$ $M_x = - P.x$

We know that differential equation of elastic line

$$EI\frac{d^2y}{dx^2} = M_x = -P.x$$

Integrating both side we get

$$\int EI\frac{d^2y}{dx^2} = -P\int x\,dx$$

$$\text{or } EI\frac{dy}{dx} = -P.\frac{x^2}{2} + A \qquad \text{............(i)}$$

Again integrating both side we get

$$EI\int dy = \int\left(P\frac{x^2}{2} + A\right)dx$$

$$\text{or } EIy = -\frac{Px^3}{6} + Ax + B \qquad \text{.............(ii)}$$

Where A and B is integration constants.

Now apply boundary condition at fixed end which is at a distance x = L from free end and we also know that at fixed end

at $x = L$, $y = 0$

at $x = L$, $\frac{dy}{dx} = 0$

from equation (ii) $EIL = -\frac{PL^3}{6} + AL + B$ ....(iii)

from equation (i) $EI.(0) = -\frac{PL^2}{2} + A$ .....(iv)

Solving (iii) & (iv) we get $A = \frac{PL^2}{2}$ and $B = -\frac{PL^3}{3}$

Therefore, $y = -\frac{Px^3}{6EI} + \frac{PL^2x}{2EI} - \frac{PL^3}{3EI}$

The slope as well as the deflection would be maximum at free end hence putting x = 0 we get

$y_{max} = -\frac{PL^3}{3EI}$ (Negative sign indicates the deflection is downward)

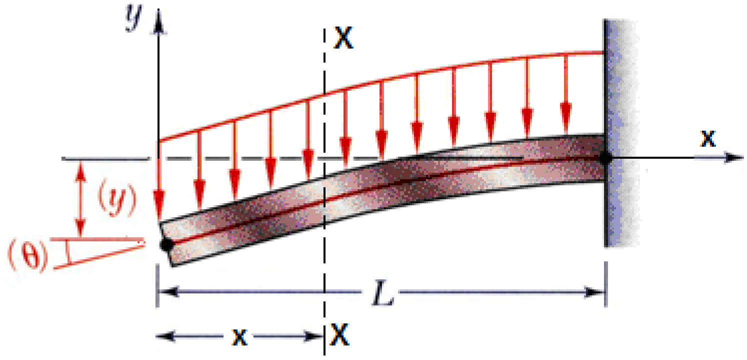
$(Slope)_{max} = \theta_{max} = \frac{PL^2}{2EI}$

Remember for a cantilever beam with a point load at free end.

Downward deflection at free end, $\boxed{(\delta) = \frac{PL^3}{3EI}}$

And slope at free end, $\boxed{(\theta) = \frac{PL^2}{2EI}}$

(ii) A Cantilever beam with UDL (uniformly distributed load)



We will now solve this problem by double integration method, for that at first we have to calculate (M_x).

Consider any section XX at a distance 'x' from free end which is left end as shown in figure.

$$\therefore M_x = -(w.x).\frac{x}{2} = -\frac{wx^2}{2}$$

We know that differential equation of elastic line

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

Integrating both sides we get

$$\int EI\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2}dx$$

$$or \quad EI\frac{dy}{dx} = -\frac{wx^3}{6} + A \quad ......(i)$$

Again integrating both side we get

$$EI\int dy = \int\left(-\frac{wx^3}{6} + A\right)dx$$

$$\text{or} \quad EIy = -\frac{wx^4}{24} + Ax + B.......(ii)$$

$$\left[\text{where A and B are integration constants}\right]$$

Now apply boundary condition at fixed end which is at a distance x = L from free end and we also know that at fixed end.

at $x = L, \quad y = 0$

at $x = L, \quad \frac{dy}{dx} = 0$

from equation (i) we get $\quad EI \times (0) = \frac{-wL^3}{6} + A \text{ or } A = \frac{+wL^3}{6}$

from equation (ii) we get $EI.y = -\dfrac{wL^4}{24} + A.L + B$

or $B = -\dfrac{wL^4}{8}$

The slope as well as the deflection would be maximum at the free end hence putting x = 0, we get

$$y_{max} = -\frac{wL^4}{8EI} \qquad \text{[Negative sign indicates the deflection is downward]}$$

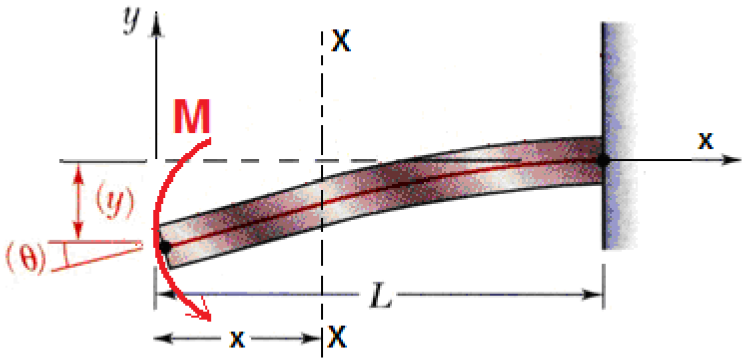
$$(\text{slope})_{max} = \theta_{max} = \frac{wL^3}{6EI}$$

Remember: For a cantilever beam with UDL over its whole length,

Maximum deflection at free end $\left(\delta\right) = \dfrac{wL^4}{8EI}$

Maximum slope, $\left(\theta\right) = \dfrac{wL^3}{6EI}$

(iii) A Cantilever beam of length 'L' with an applied moment 'M' at free end.



Consider a section XX at a distance 'x' from free end, the bending moment at section XX is

$(M_x) = -M$

We know that differential equation of elastic line or $EI\frac{d^2y}{dx^2} = -M$

Integrating both side we get

or $EI\int\frac{d^2y}{dx^2} = -\int M\,dx$

or $EI\frac{dy}{dx} = -Mx + A$ ...(i)

Again integrating both side we get

$EI\int dy = \int(Mx + A)dx$

or $EI\,y = -\frac{Mx^2}{2} + Ax + B$ ...(ii)

Where A and B are integration constants.

applying boundary conditions in equation (i) &(ii)

at $x = L$, $\frac{dy}{dx} = 0$ gives $A = ML$

at $x = L$, $y = 0$ gives $B = \frac{ML^2}{2} - ML^2 = -\frac{ML^2}{2}$

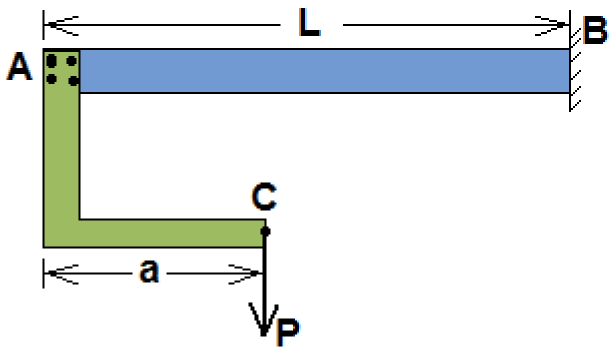
Therefore deflection equation is $y = -\frac{Mx^2}{2EI} + \frac{MLx}{EI} - \frac{ML^2}{2EI}$

Which is the equation of elastic curve.

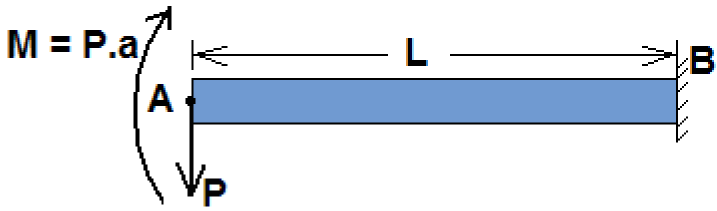
$\therefore$ Maximum deflection at free end $(\delta) = \frac{ML^2}{2EI}$ (It is downward)

$\therefore$ Maximum slope at free end $(\theta) = \frac{ML}{EI}$

Let us take a funny example: A cantilever beam AB of length 'L' and uniform flexural rigidity EI has a bracket BA (attached to its free end. A vertical downward force P is applied to free end C of the bracket. Find the ratio a/L required in order that the deflection of point A is zero. **[ISRO – 2008]**



We may consider this force 'P' and a moment (P.a) act on free end A of the cantilever beam.



Due to point load 'P' at free end 'A' downward deflection $(\delta) = \frac{PL^3}{3EI}$

Due to moment M = P.a at free end 'A' upward deflection $(\delta) = \frac{ML^2}{2EI} = \frac{(P.a)L^2}{2EI}$

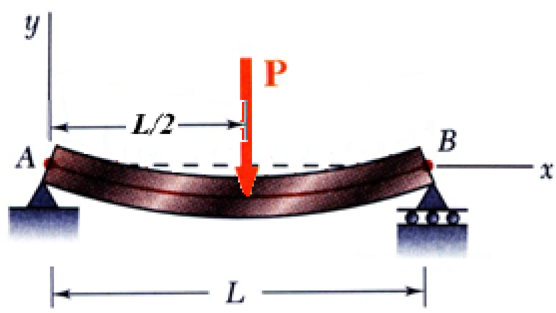
For zero deflection of free end A

$$\frac{PL^3}{3EI} = \frac{(P.a)L^2}{2EI}$$

$$\text{or } \frac{a}{L} = \frac{2}{3}$$

(iv) A simply supported beam with a point load P at its midpoint.

A simply supported beam AB carries a concentrated load P at its midpoint as shown in the figure.



We want to locate the point of maximum deflection on the elastic curve and find its value.

In the region 0 < x < L/2

Bending moment at any point x (According to the shown co-ordinate system)

$$M_x = \left(\frac{P}{2}\right).x$$

and **In the region L/2 < x < L**

$$M_x = \frac{P}{2}(x - L/2)$$

We know that differential equation of elastic line

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}.x \qquad \text{(In the region } 0 < x < L/2\text{)}$$

Integrating both side we get

$$\text{or } EI\int\frac{d^2y}{dx^2} = \int\frac{P}{2}x\,dx$$

$$\text{or } EI\frac{dy}{dx} = \frac{P}{2}.\frac{x^2}{2} + A \quad \text{(i)}$$

Again integrating both side we get

$$EI\int dy = \int\left(\frac{P}{4}x^2 + A\right)dx$$

$$\text{or } EI\,y = \frac{Px^3}{12} + Ax + B \text{ (ii)}$$

[Where A and B are integrating constants]

Now applying boundary conditions to equation (i) and (ii) we get

at $x = 0$, $y = 0$

at $x = L/2$, $\frac{dy}{dx} = 0$

$A = -\frac{PL^2}{16}$ and $B = 0$

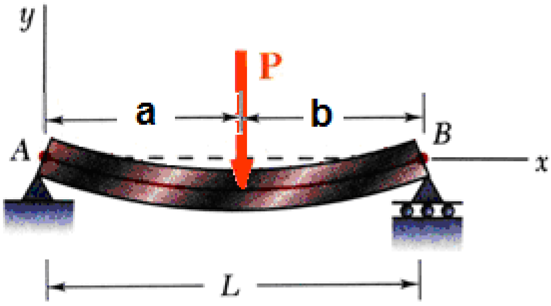
$\therefore$ Equation of elastic line, $y = \frac{Px^3}{12} - \frac{PL^{12}}{16}x$

Maximum deflection at mid span (x = L/2) $(\delta) = \frac{PL^3}{48EI}$

and maximum slope at each end $(\theta) = \dfrac{PL^2}{16EI}$

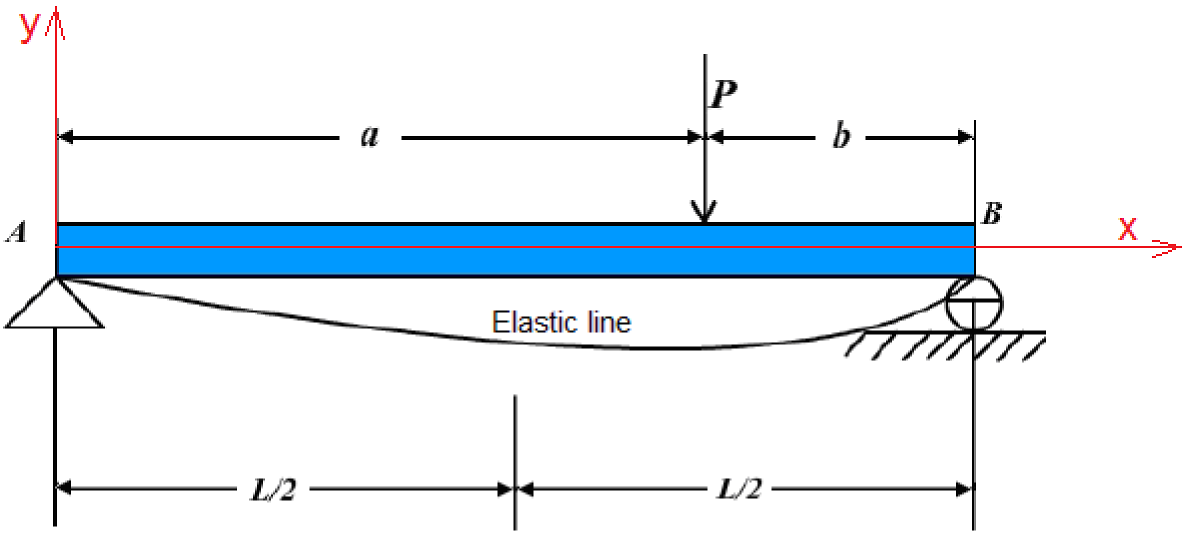
(v) A simply supported beam with a point load 'P' NOT at its midpoint.

A simply supported beam AB carries a concentrated load P as shown in the figure.



We have to locate the point of maximum deflection on the elastic curve and find the value of this deflection.

Taking co-ordinate axes x and y as shown below



For the bending moment we have

In the region $0 \le x \le a$, $M_x = \left(\dfrac{P.a}{L}\right).x$

And, In the region $a \le x \le L$, $M_x = -\dfrac{P.a}{L}(L - x)$

So we obtain two differential equation for the elastic curve.

$$EI\frac{d^2y}{dx^2} = \frac{P.a}{L}.x \qquad \text{for } 0 \le x \le a$$

and $$EI\frac{d^2y}{dx^2} = -\frac{P.a}{L}.(L - x) \qquad \text{for } a \le x \le L$$

Successive integration of these equations gives

$$EI\frac{dy}{dx} = \frac{P.a}{L}.\frac{x^2}{2} + A_1 \qquad \text{......(i)} \qquad \text{for } o \le x \le a$$

$$EI\frac{dy}{dx} = P.a\,x - \frac{P.a}{L}x^2 + A_2 \qquad \text{......(ii)} \qquad \text{for } a \le x \le L$$

$$EI\,y = \frac{P.a}{L}.\frac{x^3}{6} + A_1x + B_1 \qquad \text{......(iii)} \qquad \text{for } 0 \le x \le a$$

$$EI\,y = P.a\frac{x^2}{2} - \frac{P.a}{L}.\frac{x^3}{6} + A_2\,x + B_2 \qquad \text{.....(iv)} \qquad \text{for } a \le x \le L$$

Where A_1, A_2, B_1, B_2 are constants of Integration.

Now we have to use Boundary conditions for finding constants:

BC^S (a) at $x = 0, y = 0$

(b) at $x = L, y = 0$

(c) at $x = a$, $\left(\frac{dy}{dx}\right)$ = Same for equation (i) & (ii)

(d) at $x = a$, y = same from equation (iii) & (iv)

We get $$A_1 = \frac{Pb}{6L}\left(L^2 - b^2\right); \qquad A_2 = \frac{P.a}{6L}\left(2L^2 + a^2\right)$$

and $$B_1 = 0; \qquad B_2 = Pa^3 / 6EI$$

Therefore we get two equations of elastic curve

$$EI\,y = -\frac{Pbx}{6L}\left(L^2 - b^2 - x^2\right) \qquad \text{..... (v)} \qquad \text{for } 0 \le x \le a$$

$$EI\,y = \frac{Pb}{6L}\left[\left(\frac{L}{b}\right)(x - a)^3 + \left(L^2 - b^2\right)x - x^3\right] \qquad \text{. ...(vi)} \qquad \text{for } a \le x \le L$$

For a > b, the maximum deflection will occur in the left portion of the span, to which equation (v) applies. Setting the derivative of this expression equal to zero gives

$$x = \sqrt{\frac{a(a+2b)}{3}} = \sqrt{\frac{(L-b)(L+b)}{3}} = \sqrt{\frac{L^2 - b^2}{3}}$$

at that point a horizontal tangent and hence the point of maximum deflection substituting this value of x into equation (v), we find, $y_{max} = \dfrac{P.b(L^2 - b^2)^{3/2}}{9\sqrt{3}.\ EIL}$

Case –I: if a = b = L/2 then

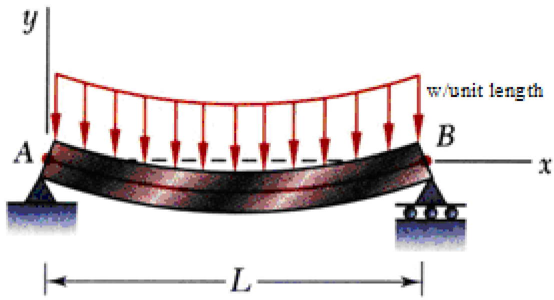
Maximum deflection will be at $x = \sqrt{\dfrac{L^2 - (L/2)^2}{3}} = L/2$

i.e. at mid point

and $y_{max} = (\delta) = \dfrac{P.(L/2)\times\left\{L^2 - (L/2)^2\right\}^{3/2}}{9\sqrt{3}\,EIL} = \dfrac{PL^3}{48EI}$

(vi) A simply supported beam with UDL (Uniformly distributed load)

A simply supported beam AB carries a uniformly distributed load (UDL) of intensity w/unit length over its whole span L as shown in figure. We want to develop the equation of the elastic curve and find the maximum deflection δ at the middle of the span.



Taking co-ordinate axes x and y as shown, we have for the bending moment at any point x

$$M_x = \frac{wL}{2}.x - w.\frac{x^2}{2}$$

Then the differential equation of deflection becomes

$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{2}.x - w.\frac{x^2}{2}$$

Integrating both sides we get

$$EI\frac{dy}{dx} = \frac{wL}{2}.\frac{x^2}{2} - \frac{w}{2}.\frac{x^3}{3} + A \qquad .....(i)$$

Again Integrating both side we get

$$EI\,y = \frac{wL}{2}.\frac{x^3}{6} - \frac{w}{2}.\frac{x^4}{12} + Ax + B \qquad .....(ii)$$

Where A and B are integration constants. To evaluate these constants we have to use boundary conditions.

at x = 0, y = 0 gives B = 0

at x = L/2, $\frac{dy}{dx} = 0$ gives $A = -\frac{wL^3}{24}$

Therefore the equation of the elastic curve

$$y = \frac{wL}{12EI}.x^3 - \frac{w}{24EI}.x^4 - \frac{wL^3}{12EI}.x = \frac{wx}{24EI}\left[L^3 - 2L.x^2 + x^3\right]$$

The maximum deflection at the mid-span, we have to put x = L/2 in the equation and obtain

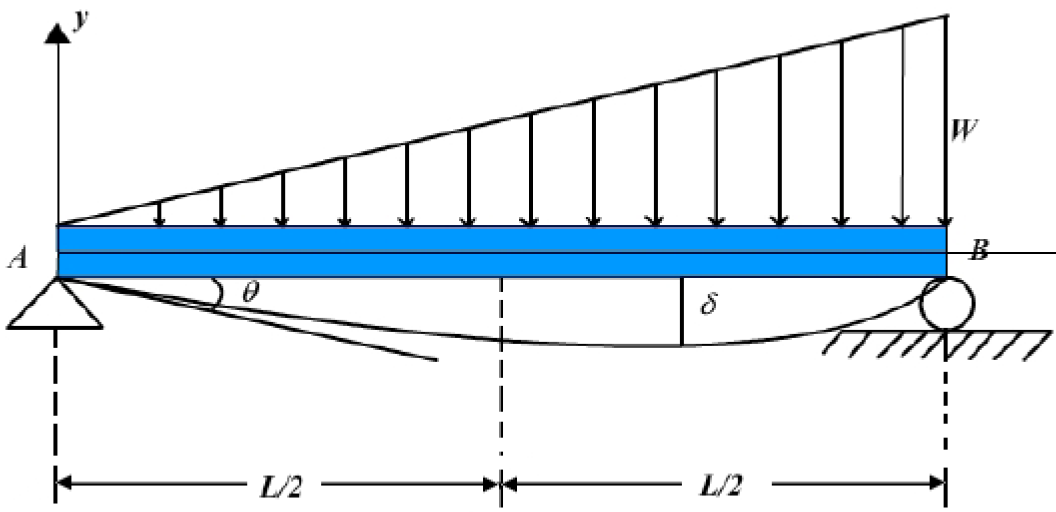
Maximum deflection at mid-span, $\boxed{(\delta) = \frac{5wL^4}{384EI}}$ (It is downward)

And Maximum slope $\theta_A = \theta_B$ at the left end A and at the right end b is same putting x = 0 or x = L

Therefore we get Maximum slope $\boxed{(\theta) = \frac{wL^3}{24EI}}$

(vii) A simply supported beam with triangular distributed load (GVL) gradually varied load.

A simply supported beam carries a triangular distributed load (GVL) as shown in figure below. We have to find equation of elastic curve and find maximum deflection (δ).



In this (GVL) condition, we get

$$EI\,\frac{d^4y}{dx^4} = load = -\frac{w}{L}.x \qquad .....(i)$$

Separating variables and integrating we get

$$EI\,\frac{d^3y}{dx^3} = \left(V_x\right) = -\frac{wx^2}{2L} + A \qquad .....(ii)$$

Again integrating thrice we get

$$EI\,\frac{d^2y}{dx^2} = M_x = -\frac{wx^3}{6L} + Ax + B \qquad .....(iii)$$

$$EI\,\frac{dy}{dx} = -\frac{wx^4}{24L} + \frac{Ax^2}{2} + Bx + C \qquad .....(iv)$$

$$EI\,y = -\frac{wx^5}{120L} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \qquad .....(v)$$

Where A, B, C and D are integration constant.

Boundary conditions at x = 0, $M_x = 0$, y = 0

at x = L, $M_x = 0$, y = 0 gives

$A = \frac{wL}{6}$, B = 0, $C = -\frac{7wL^3}{360}$, D = 0

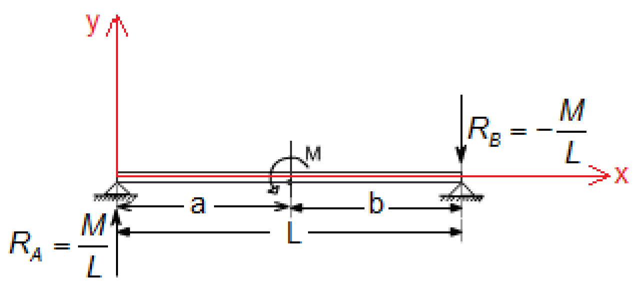
Therefore $y = -\frac{wx}{360EIL}\left\{7L^4 - 10L^2x^2 + 3x^4\right\}$ *(negative sign indicates downward deflection)*

To find maximum deflection δ, we have $\frac{dy}{dx} = 0$

And it gives x = 0.519 L and maximum deflection $(\delta) = 0.00652\frac{wL^4}{EI}$

(viii) A simply supported beam with a moment at mid-span.

A simply supported beam AB is acted upon by a couple M applied at an intermediate point distance 'a' from the equation of elastic curve and deflection at point where the moment acted.



Considering equilibrium we get $R_A = \frac{M}{L}$ and $R_B = -\frac{M}{L}$

Taking co-ordinate axes x and y as shown, we have for bending moment

In the region $0 \le x \le a$, $M_x = \frac{M}{L}.x$

In the region $a \le x \le L$, $M_x = \frac{M}{L}x - M$

So we obtain the difference equation for the elastic curve

$$EI\frac{d^2y}{dx^2} = \frac{M}{L}.x \qquad \text{for } 0 \le x \le a$$

$$\text{and } EI\frac{d^2y}{dx^2} = \frac{M}{L}.x - M \qquad \text{for } a \le x \le L$$

Successive integration of these equation gives

$$EI\frac{dy}{dx} = \frac{M}{L}.\frac{x^2}{2} + A_1 \qquad ....(i) \qquad \text{for } 0 \le x \le a$$

$$EI\frac{dy}{dx} = \frac{M}{L} = \frac{x^2}{2} - Mx + A_2 \qquad .....(ii) \qquad \text{for } a \le x \le L$$

$$\text{and } EI\,y = \frac{M}{L}.\frac{x^3}{\sigma} + A_1x + B_1 \qquad ......(iii) \qquad \text{for } 0 \le x \le a$$

$$EI\,y = \frac{M}{L}\frac{x^3}{\sigma} - \frac{Mx^2}{2} + A_2x + B_2 \qquad ......(iv) \qquad \text{for } a \le x \le L$$

Where A_1, A_2, B_1 and B_2 are integration constants.

To finding these constants boundary conditions

(a) at $x = 0$, $y = 0$

(b) at $x = L$, $y = 0$

(c) at $x = a$, $\left(\frac{dy}{dx}\right)$ = same form equation (i) & (ii)

(d) at $x = a$, y = same form equation (iii) & (iv)

$$A_1 = -M.a + \frac{ML}{3} + \frac{Ma^2}{2L}, \quad A_2 = \frac{ML}{3} + \frac{Ma^2}{2L}$$

$$B_1 = 0, \qquad B_2 = \frac{Ma^2}{2}$$

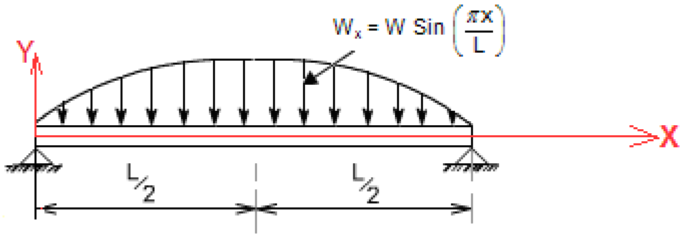
With this value we get the equation of elastic curve

$$y = -\frac{Mx}{6L}\left\{6aL - 3a^2 - x^2 - 2L^2\right\} \qquad \text{for } 0 \le x \le a$$

$\therefore$ deflection of x = a,

$$y = \frac{Ma}{3EIL}\left\{3aL - 2a^2 - L^2\right\}$$

(ix) A simply supported beam with a continuously distributed load the intensity of which at any point 'x' along the beam is $w_x = w\sin\left(\frac{\pi x}{L}\right)$



At first we have to find out the bending moment at any point 'x' according to the shown co-ordinate system.

We know that

$$\frac{d(V_x)}{dx} = -w\sin\left(\frac{\pi x}{L}\right)$$

Integrating both sides we get

$$\int d(V_x) = -\int w\ \sin\left(\frac{\pi x}{L}\right)dx + A$$

$$\text{or } V_x = +\frac{wL}{\pi}.\cos\left(\frac{\pi x}{L}\right) + A$$

and we also know that

$$\frac{d(M_x)}{dx} = V_x = \frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) + A$$

Again integrating both sides we get

$$\int d(M_x) = \int \left\{\frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) + A\right\}dx$$

$$\text{or } M_x = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right) + Ax + B$$

Where A and B are integration constants, to find out the values of A and B. We have to use boundary conditions

at $x = 0$, $M_x = 0$

and at $x = L$, $M_x = 0$

From these we get A = B = 0. Therefore $M_x = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right)$

So the differential equation of elastic curve

$$EI\frac{d^2y}{dx^2}=M_x=\frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right)$$

Successive integration gives

$$EI\frac{dy}{dx}=-\frac{wL^3}{\pi^3}\cos\left(\frac{\pi x}{L}\right)+C \qquad ....(i)$$

$$EIy=-\frac{wL^4}{\pi^4}\sin\left(\frac{\pi x}{L}\right)+Cx+D \qquad .....(ii)$$

Where C and D are integration constants, to find out C and D we have to use boundary conditions

at $x = 0$, $y = 0$

at $x = L$, $y = 0$

and that give $C = D = 0$

Therefore slope equation $EI\frac{dy}{dx}=-\frac{wL^3}{\pi^3}\cos\left(\frac{\pi x}{L}\right)$

and Equation of elastic curve $y=-\frac{wL^4}{\pi^4 EI}\sin\left(\frac{\pi x}{L}\right)$

(-ive sign indicates deflection is downward)

Deflection will be maximum if $\sin\left(\frac{\pi x}{L}\right)$ is maximum

$\sin\left(\frac{\pi x}{L}\right)=1$ or $x = L/2$

and Maximum downward deflection $(\delta)=\frac{WL^4}{\pi^4 EI}$ (downward).

5.5 Macaulay's Method (Use of singularity function)

- When the beam is subjected to point loads (but several loads) this is very convenient method for determining the deflection of the beam.
- In this method we will write single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.
- After integrating this equation we will find the integration constants which are valid for entire length of the beam. This method is known as **method of singularity constant.**

Procedure to solve the problem by Macaulay's method

Step – I: Calculate all reactions and moments

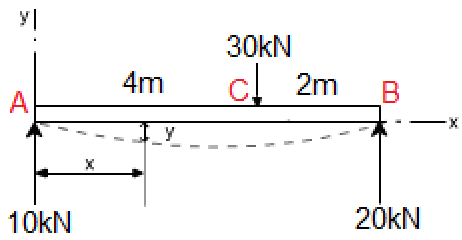
Step – II: Write down the moment equation which is valid for all values of x. This must contain brackets.

Step – III: Integrate the moment equation by a typical manner. Integration of (x-a) will be $\frac{(x\text{-}a)^2}{2}$ not $\left(\frac{x^2}{2} - ax\right)$ and integration of $(x\text{-}a)^2$ will be $\frac{(x\text{-}a)^3}{3}$ so on.

Step – IV: After first integration write the first integration constant (A) after first terms and after second time integration write the second integration constant (B) after A.x . Constant A and B are valid for all values of x.

Step – V: Using Boundary condition find A and B at a point x = p if any term in Macaulay's method, (x-a) is negative (-ive) the term will be neglected.

(i) Let us take an example: A simply supported beam AB length 6m with a point load of 30 kN is applied at a distance 4m from left end A. Determine the equations of the elastic curve between each change of load point and the maximum deflection of the beam.



Answer: We solve this problem using Macaulay's method, for that first writes the general momentum equation for the last portion of beam BC of the loaded beam.

$$EI\,\frac{d^2y}{dx^2} = M_x = 10x \;\left|-30(x-4)\right| \;\; N.m \qquad ....(i)$$

By successive integration of this equation (using Macaulay's integration rule

e.g $\int (x-a)\,dx = \frac{(x-a)^2}{2}$)

We get

$$EI\,\frac{dy}{dx} = 5x^2 + A \;\left|-15(x\text{-}4)^2\right| \;\; N.m^2 \qquad .....(ii)$$

$$\text{and}\;\; EI\,y = \frac{5}{3}x^3 + Ax + B \;\left|-5\,(x-4)^3\right| \; N.m^3 \qquad .....(iii)$$

Where A and B are two integration constants. To evaluate its value we have to use following boundary conditions.

at x = 0, y = 0

and at x = 6m, y = 0

Note: When we put x = 0, x - 4 is negativre (–ive) and this term will ***not*** be considered for x = 0 , so our equation will be EI y = $\frac{5}{3}x^3 + Ax + B$, and at x = 0 , y = 0 gives B = 0

But when we put x = 6, x-4 is positive (+ive) and this term will be considered for x = 6, y = 0 so our equation will be EI y = $\frac{5}{3}x^3 + Ax + 0 - 5(x-4)^3$

This gives

$$EI.(0) = \frac{5}{3}.6^3 + A.6 + 0 - 5(6-4)^3$$

or A = - 53

So our slope and deflection equation will be

$$EI\frac{dy}{dx} = 5x^2 - 53 \left| -15(x-4)^2 \right|$$

$$\text{and } EI\,y = \frac{5}{3}x^3 - 53x + 0 \left| -5(x-4)^3 \right|$$

Now we have two equations for entire section of the beam and we have to understand how we use these equations. Here if x < 4 then x – 4 is negative so this term will be deleted. That so why in the region $o \le x \le 4m$ we will neglect (x – 4) term and our slope and deflection equation will be

$$EI\frac{dy}{dx} = 5x^2 - 53$$

$$\text{and} \quad EI\,y = \frac{5}{3}x^3 - 53x$$

But in the region $4m < x \le 6m$, (x – 4) is positive so we include this term and our slope and deflection equation will be

$$EI\frac{dy}{dx} = 5x^2 - 53 - 15(x-4)^2$$

$$EI\,y = \frac{5}{3}x^3 - 53x - 5(x-4)^3$$

Now we have to find out maximum deflection, but we don't know at what value of 'x' it will be maximum. For this assuming the value of 'x' will be in the region $0 \le x \le 4m$.

Deflection (y) will be maximum for that $\frac{dy}{dx} = 0$ or $5x^2 - 53 = 0$ or x = 3.25 m as our calculated x is in the region $0 \le x \le 4m$; at x = 3.25 m deflection will be maximum

$$\text{or} \quad EI\,y_{max} = \frac{5}{3} \times 3.25^3 - 53 \times 3.25$$

or $\quad y_{max} = -\dfrac{115}{EI}$ $\quad$ (-ive sign indicates downward deflection)

But if you have any doubt that Maximum deflection may be in the range of $4 < x \le 6m$, use EIy = $5x^2 - 53x - 5\,(x-4)^3$ and find out x. The value of x will be absurd that indicates the maximum deflection will not occur in the region $4 < x \le 6m$.

Deflection (y) will be maximum for that $\dfrac{dy}{dx} = 0$

or $\quad 5x^2 - 53 - 15(x-4)^2 = 0$

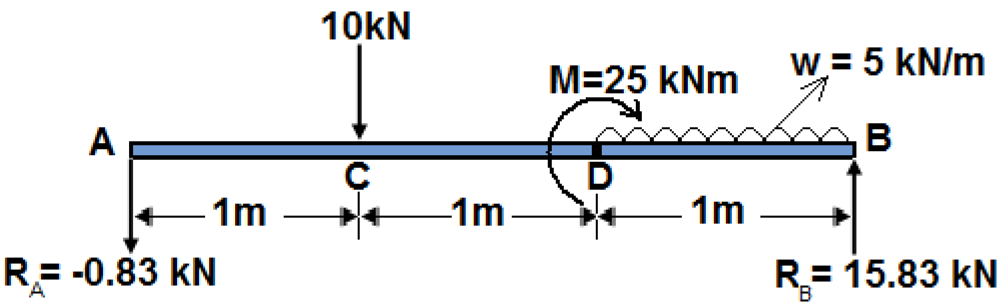
or $\quad 10x^2 - 120x + 293 = 0$

or $\quad x = 3.41$ m or 8.6 m

Both the value fall outside the region $4 < x \le 6m$ and in this region $4 < x \le 6m$ and in this region maximum deflection will not occur.

(ii) Now take an example where Point load, UDL and Moment applied simultaneously in a beam:

Let us consider a simply supported beam AB (see Figure) of length 3m is subjected to a point load 10 kN, UDL = 5 kN/m and a bending moment M = 25 kNm. Find the deflection of the beam at point D if flexural rigidity (EI) = 50 KNm^2.



Answer: Considering equilibrium

$\sum M_A = 0$ gives

$-10 \times 1 - 25 - (5\times1)\times(1+1+1/2) + R_B \times 3 = 0$

or $R_B = 15.83\,kN$

$R_A + R_B = 10 + 5\times1$ gives $R_A = -0.83\,kN$

We solve this problem using Macaulay's method, for that first writing the general momentum equation for the last portion of beam, DB of the loaded beam.

$$EI\,\frac{d^2y}{dx^2} = M_x = -0.83x \quad \Big| -10(x-1) \Big| +25(x-2) \Big| -\frac{5(x-2)^2}{2} \Big|$$

By successive integration of this equation (using Macaulay's integration rule

e.g $\int (x-a)\,dx = \frac{(x-a)^2}{2}$)

We get

$$EI\,\frac{dy}{dx} = -\frac{0.83}{2}.x^2 + A \; \Big| -5(x-1)^2 \Big| +25(x-2) \Big| -\frac{5}{6}(x-2)^3 \Big|$$

$$\text{and } EIy = -\frac{0.83}{6}x^3 + Ax + B \; \Big| -\frac{5}{3}(x-1)^3 \Big| +\frac{25}{2}(x-2)^2 \Big| -\frac{5}{24}(x-2)^4 \Big|$$

Where A and B are integration constant we have to use following boundary conditions to find out A & B.

at $x = 0$, $y = 0$

at $x = 3m$, $y = 0$

Therefore $B = 0$

$$\text{and } 0 = -\frac{0.83}{6} \times 3^3 + A \times 3 + 0 \; \Big| -\frac{5}{3} \times 2^3 \Big| +12.5 \times 1^2 \Big| -\frac{5}{24} \times 1^4 \Big|$$

or $A = 1.93$

$$EIy = -0.138x^3 + 1.93x \; \Big| -1.67(x-1)^3 \Big| +12.5(x-2)^2 \Big| -0.21(x-2)^4 \Big|$$

Deflextion at point D at x = 2m

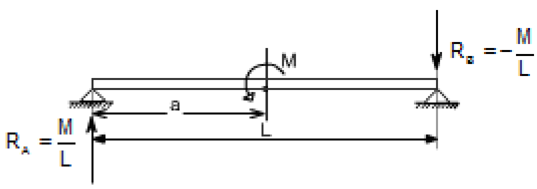
$$EIy_D = -0.138 \times 2^3 + 1.93 \times 2 - 1.67 \times 1^3 = -8.85$$

$$\text{or } y_D = -\frac{8.85}{EI} = -\frac{8.85}{50 \times 10^3}\text{m} \quad (-\text{ive sign indicates deflection downward})$$

$$= 0.177\text{mm}(\text{downward}).$$

(iii) A simply supported beam with a couple M at a distance 'a' from left end

If a couple acts we have to take the distance in the bracket and this should be raised to the power zero. i.e. $M(x - a)^0$. Power is zero because $(x - a)^0 = 1$ and unit of $M(x - a)^0 = M$ but we introduced the distance which is needed for Macaulay's method.



$$EI\,\frac{d^2y}{dx^2} = M = R_A x - M(x\text{-}a)^0$$

Successive integration gives

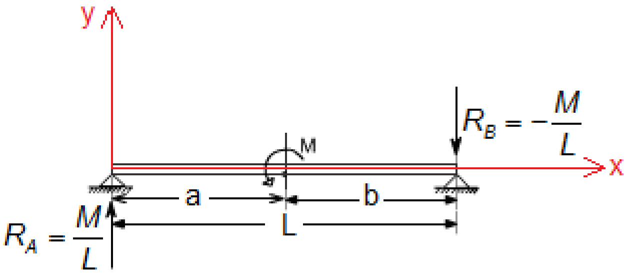
$$EI\frac{dy}{dx}=\frac{M}{L}.\frac{x^2}{2}+A-M(x\text{-}a)^1$$

$$EI\,y=\frac{M}{6L}x^3+Ax+B-\frac{M(x\text{-}a)^2}{2}$$

Where A and B are integration constants, we have to use boundary conditions to find out A & B.

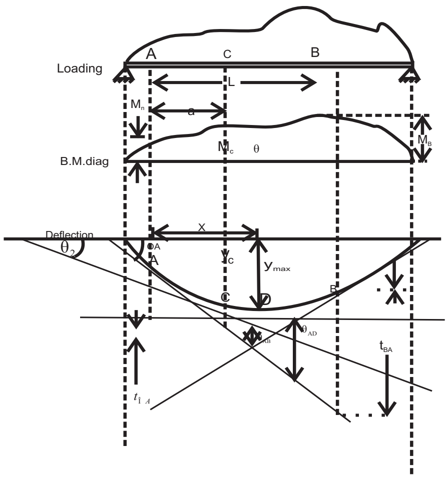
at x = 0, y = 0 gives B = 0

at x = L, y = 0 gives $A=\frac{M(L\text{-}a)^2}{2L}-\frac{ML}{6}$



8. Moment area method

- This method is used generally to obtain displacement and rotation at a single point on a beam.
- The moment area method is convenient in case of beams acted upon with point loads in which case bending moment area consist of triangle and rectangles.



- ***Angle between the tangents drawn at 2 points A&B on the elastic line,*** θ_{AB}

$$\theta_{AB}=\frac{1}{EI}\times\text{Area of the bending moment diagram between A\&B}$$

i.e. slope $\boxed{\theta_{AB}=\frac{A_{B.M.}}{EI}}$

- ***Deflection of B related to 'A'***

y_{BA} = **Moment of** $\dfrac{M}{EI}$ diagram between B&A taking about B (or w.r.t. **B)**

i.e. deflection $y_{BA} = \dfrac{A_{B.M} \times \bar{x}}{EI}$

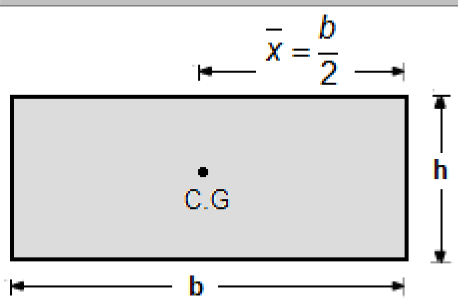
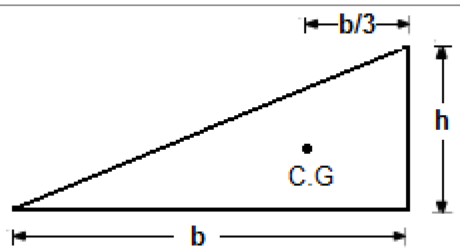
Important Note

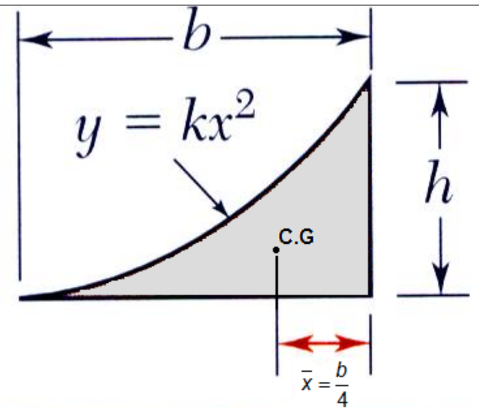
If A_1 = Area of shear force (SF) diagram

A_2 = Area of bending moment (BM) diagram,

Then, Change of slope over any portion of the loaded beam = $\dfrac{A_1 \times A_2}{EI}$

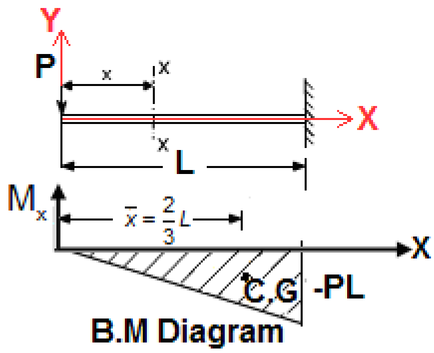
Some typical bending moment diagram and their area (A) and distance of C.G from one edge $(\bar{x})$ is shown in the following table. [Note the distance will be different from other end]

Shape	BM Diagram	Area	Distance from C.G
1. Rectangle		$A = bh$	$\bar{x} = \dfrac{b}{2}$
2. Triangle			$\bar{x} = \dfrac{b}{2}$

3. Parabola			$\bar{x} = \frac{b}{4}$
4. Parabola			
5.Cubic Parabola			
6. y = k x^n			
7. Sine curve			

Determination of Maximum slope and deflection by Moment Area- Method

(i) A Cantilever beam with a point load at free end



Area of BM (Bending moment diagram)

$$(A) = \frac{1}{2} \times L \times PL = \frac{PL^2}{2}$$

Therefore

$$\text{Maximum slope}(\theta) = \frac{A}{EI} = \frac{PL^2}{2EI} \qquad \text{(at free end)}$$

$$\text{Maximum deflection } (\delta) = \frac{A\bar{x}}{EI}$$

$$= \frac{\left(\frac{PL^2}{2}\right) \times \left(\frac{2}{3}L\right)}{EI} = \frac{PL^3}{3EI} \qquad \text{(at free end)}$$

(ii) A cantilever beam with a point load *not* at free end

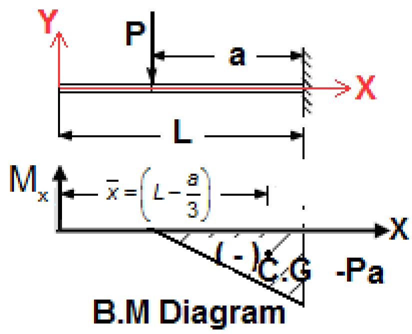
Area of BM diagram $(A) = \frac{1}{2} \times a \times Pa = \frac{Pa^2}{2}$

Therefore

Maximum slope $(\theta) = \frac{A}{EI} = \frac{Pa^2}{2EI}$ (at free end)

Maximum deflection $(\delta) = \frac{A\bar{x}}{EI}$

$$= \frac{\left(\frac{Pa^2}{2}\right) \times \left(L - \frac{a}{3}\right)}{EI} = \frac{Pa^2}{2EI} \cdot \left(L - \frac{a}{3}\right) \quad \text{(at free end)}$$



(iii) A cantilever beam with UDL over its whole length

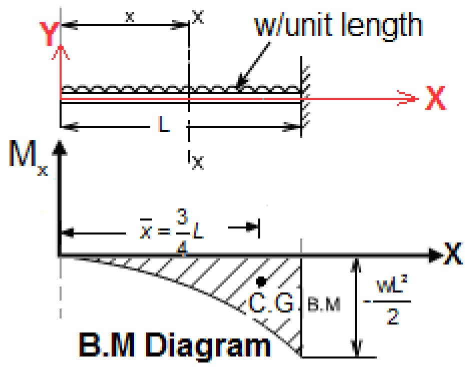
Area of BM diagram $(A) = \frac{1}{3} \times L = \left(\frac{wL^2}{2}\right) = \frac{wL^3}{6}$

Therefore

Maximum slope $(\theta) = \frac{A}{EI} = \frac{wL^3}{6EI}$ (at free end)

Maximum deflection $(\delta) = \frac{A\bar{x}}{EI}$

$$= \frac{\left(\frac{wL^3}{6}\right) \times \left(\frac{3}{4}L\right)}{EI} = \frac{wL^4}{8EI} \quad \text{(at free end)}$$



(iv) A simply supported beam with point load at mid-spam

Area of shaded BM diagram

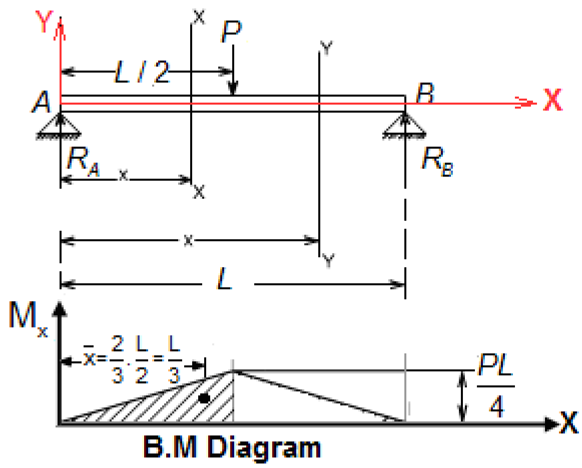
$$(A) = \frac{1}{2} \times \frac{L}{2} \times \frac{PL}{4} = \frac{PL^2}{16}$$

Therefore

Maximum slope $(\theta) = \frac{A}{EI} = \frac{PL^2}{16EI}$ (at each ends)

Maximum deflection $(\delta) = \frac{A\bar{x}}{EI}$

$$= \frac{\left(\frac{PL^2}{16} \times \frac{L}{3}\right)}{EI} = \frac{PL^3}{48EI} \quad \text{(at mid point)}$$



(v) A simply supported beam with UDL over its whole length

Area of BM diagram (shaded)

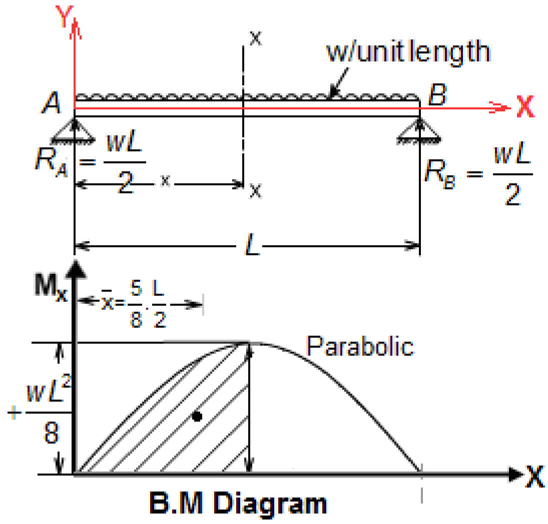
$$(A)=\frac{2}{3}\times\left(\frac{L}{2}\right)\times\left(\frac{wL^2}{8}\right)=\frac{wL^3}{24}$$

Therefore

Maximum slope $(\theta)=\dfrac{A}{EI}=\dfrac{wL^3}{24EI}$ (at each ends)

Maximum deflection $(\delta)=\dfrac{A\bar{x}}{EI}$

$$=\frac{\left(\frac{wL^3}{24}\right)\times\left(\frac{5}{8}\times\frac{L}{2}\right)}{EI}=\frac{5}{384}\frac{wL^4}{EI}$$ (at mid point)



9. Method of superposition

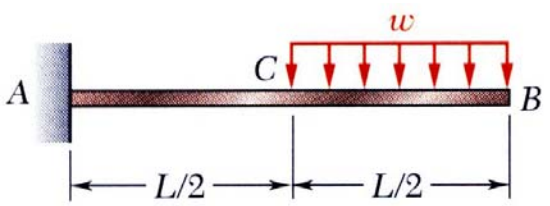
Assumptions:

- Structure should be linear
- Slope of elastic line should be very small.
- The deflection of the beam should be small such that the effect due to the shaft or rotation of the line of action of the load is neglected.

Principle of Superposition:

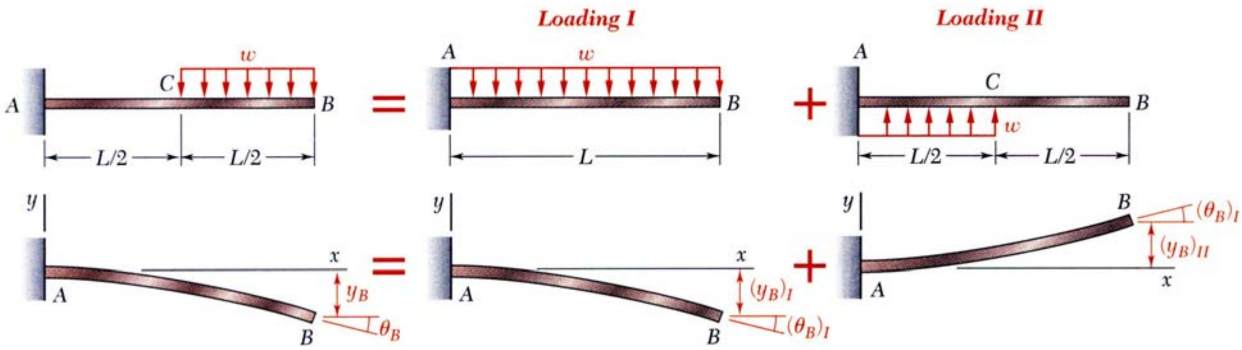
- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

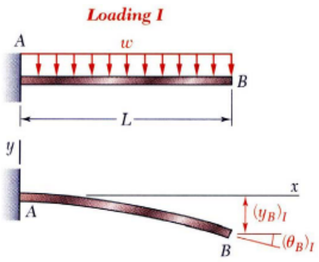
Example:



For the beam and loading shown, determine the slope and deflection at point *B*.

Superpose the deformations due to *Loading I* and *Loading II* as shown.



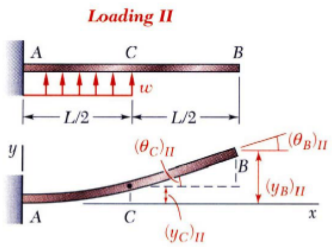


Loading I

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

Loading II

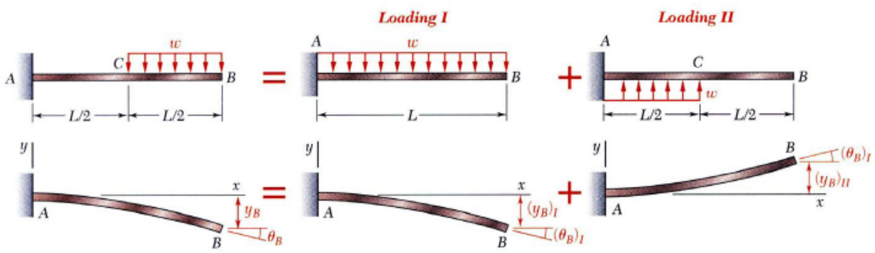
$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$



In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$



Combine the two solutions,

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI} \qquad \boxed{\theta_B = \frac{7wL^3}{48EI}}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI} \qquad \boxed{y_B = \frac{41wL^4}{384EI}}$$

10. Conjugate beam method

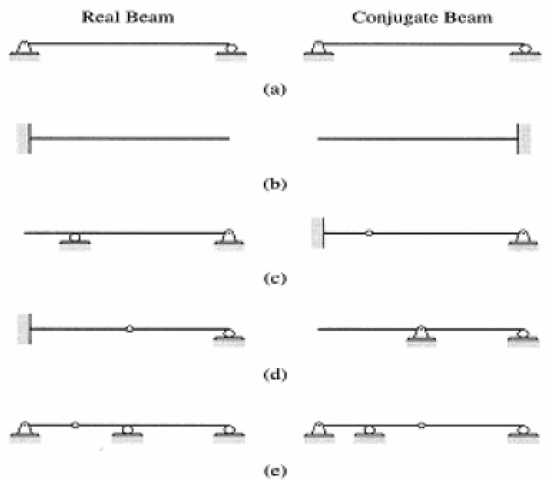
In the *conjugate beam method*, the **length** of the conjugate beam is the same as the length of the actual beam, the **loading diagram** (showing the loads acting) on the conjugate beam is simply the bending-moment diagram of the actual beam divided by the flexural rigidity *EI* of the actual beam, and the **corresponding support condition** for the conjugate beam is given by the rules as shown below.

Corresponding support condition for the conjugate beam

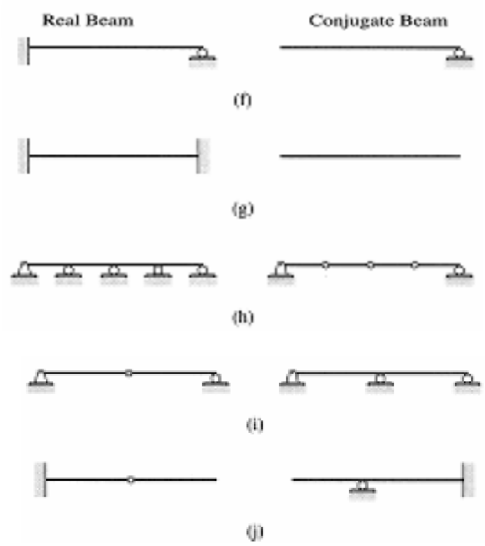
	Existing support condition of the **actual beam**	Corresponding support condition for the **conjugate beam**
Rule 1	Fixed end	Free end
Rule 2	Free end	Fixed end
Rule 3	Simple support at the end	Simple support at the end
Rule 4	Simple support *not* at the end	Unsupported hinge
Rule 5	Unsupported hinge	Simple support

Conjugates of Common Types of Real Beams

Conjugate beams for statically determinate real beams



Conjugate beams for Statically indeterminate real beams



By the conjugate beam method, the slope and deflection of the actual beam can be found by using the following two rules:

- The **slope** of the actual beam at any cross section is equal to the **shearing force** at the corresponding cross section of the conjugate beam.
- The **deflection** of the actual beam at any point is equal to the **bending moment** of the conjugate beam at the corresponding point.

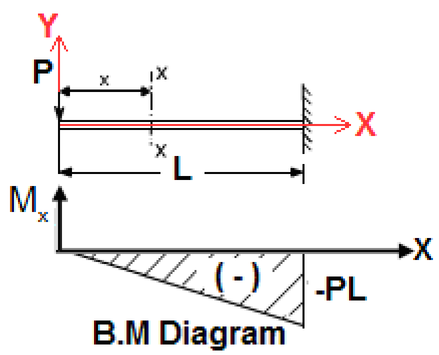
Procedure for Analysis

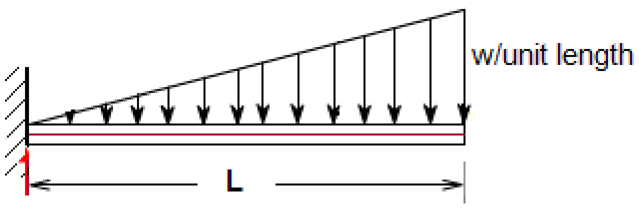
- Construct the **M / EI** diagram for the given (real) beam subjected to the specified (real) loading. If a combination of loading exists, you may use M-diagram by parts
- Determine the ***conjugate*** beam corresponding to the given real beam
- Apply the M / EI diagram as the **load on the conjugate** beam as per sign convention
- Calculate the **reactions** at the supports of the **conjugate** beam by applying equations of equilibrium and conditions
- Determine the ***shears*** in the ***conjugate*** beam at locations where **slopes** is desired in the **real** beam, $V_{conj} = \theta_{real}$
- Determine the ***bending moments*** in the ***conjugate*** beam at locations where **deflections** is desired in the **real** beam, $M_{conj} = y_{real}$

The method of double integration, method of superposition, moment-area theorems, and Castigliano's theorem are all well established methods for finding deflections of beams, but they require that the **boundary conditions** of the beams be known or specified. If not, all of them become *helpless*. However, the conjugate beam method is able to proceed and yield a solution for the possible deflections of the beam based on the **support conditions**, rather than the boundary conditions, of the beams.

(i) A Cantilever beam with a point load 'P' at its free end.

For Real Beam: At a section a distance 'x' from free end consider the forces to the left. Taking moments about the section gives (obviously to the left of the section) $M_x = -P.x$ (negative sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as negative according to the sign convention) so that the maximum bending moment occurs at the fixed end i.e. $M_{max} = -PL$ (at x = L)





Considering equilibrium we get, $M_A = \frac{wL^2}{3}$ and Reaction $(R_A) = \frac{wL}{2}$

Considering any cross-section XX which is at a distance of x from the fixed end.

At this point load $(W_x) = \frac{W}{L}.x$

Shear force $\left(V_x\right) = R_A - $ area of triangle ANM

$$= \frac{wL}{2} - \frac{1}{2}.\left(\frac{w}{L}.x\right).x = +\frac{wL}{2} - \frac{wx^2}{2L}$$

$\therefore$ The shear force variation is parabolic.

at $x = 0$, $V_x = +\frac{wL}{2}$ i.e. Maximum shear force, $V_{max} = +\frac{wL}{2}$

at $x = L$, $V_x = 0$

Bending moment $\left(M_x\right) = R_A.x - \frac{wx^2}{2L}.\frac{2x}{3} - M_A$

$$= \frac{wL}{2}.x - \frac{wx^3}{6L} - \frac{wL^2}{3}$$

$\therefore$ The bending moment variation is cubic

at $x = 0$, $M_x = -\frac{wL^2}{3}$ i.e.Maximum B.M. $\left(M_{max}\right) = -\frac{wL^2}{3}$.

at $x = L$, $M_x = 0$